Fluctuation relations for a classical harmonic oscillator in an electromagnetic field

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In this work, we establish some fluctuation relations for a classical two-dimensional system of independent charged harmonic oscillators in the presence of an electromagnetic field. The main fluctuation relation quantifies irreversible behavior by comparing probabilities of observing particular trajectories during forward and backward processes and is expressed in terms of the work performed by the externally time-dependent electric field when the system is driven away from equilibrium. In the absence of a harmonic force and assuming a constant electric field, our theoretical results reduce to the fluctuation relations for a classical two-dimensional system of noninteracting electrons under the influence of externally crossed electric and magnetic fields, which were recently studied [D. Roy and N. Kumar, Phys. Rev. E **78**, 052102 (2008)]. Finally, by considering the dragging of the center of the harmonic trap potential given by the presence of the arbitrary time-dependent electric field, the work-fluctuation theorem and the Jarzynski equality are verified.

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I. INTRODUCTION

It is known that equilibrium statistical mechanics is a well established theory because it can explain the properties of a broad variety of systems in equilibrium. Linear irreversible thermodynamics is an extension of the concepts of equilibrium thermodynamics to systems that are close to equilibrium [1–5]. These traditional concepts are limited in application to large systems or averages over an ensemble of states, referred to as the thermodynamic limit. In the last 15 years, there has been considerable interest in the study of nonequilibrium statistical mechanics of small systems, which has led to the discovery of several rigorous theorems called fluctuation theorems and related research [6–33]. These theorems explain how macroscopic irreversibility appears naturally in systems that obey time-reversible microscopic dynamics and they are essential for the application of thermodynamic concepts to nanosized systems that are of interest to biologist, physicist, and engineers. Our main purpose in this work is to obtain some fluctuation relations for the probability densities of a two-dimensional system of noninteracting charged harmonic oscillators under the influence of an electromagnetic field through the explicit solution of the Smoluchowski equation associated with the overdamped Langevin equation. The magnetic field is considered as a constant vector (pointing along the z axis) and the electric field is homogeneous but in general a time-dependent vector. The main fluctuation relation will be formulated in terms of the joint probability density $f(x,y,t)$ by relating the ratio $f(x,y,t)/f(-x,-y,t)$ to the work performed by the externally time-dependent electric field consistently with the fluctuation theorems [18,19]. The other two fluctuation relations will be given in terms of the marginal probabilities $f(x,t)$ and $f(y,t)$, which are calculated through the integration of the joint probability density $f(x,y,t)$ with respect to the y and x variables, respectively. It will be shown that, in the absence of the harmonic force, the fluctuation relations associated with $f(x,t)$ and $f(y,t)$ reduce, respectively, to those recently established by Roy and Kumar [17] for a two-dimensional system of noninteracting electrons under the externally applied crossed electric and magnetic fields. The Roy-Kumar fluctuation relation associated with $f(x,t)$ is named as longitudinal or barotropic-type fluctuation relation and that associated with $f(y,t)$ as the transverse or "Hall-fluctuation" relation. The literature on the fluctuation relations is vast and the following reviews may be useful to the reader interested in having a deeper look on them [25–27].

The other interesting result used to estimate an equilibrium thermodynamic quantity from nonequilibrium measurements is the Jarzynski equality (JE) [11], which relates the change in the free energy between two equilibrium states to an ensemble average of the work performed on the system; the average is taken between the different realizations of trajectories between the two equilibrium states. Actually, the JE has been tested by mechanically stretching a single molecule of RNA reversibly and irreversibly between two conformations [28] and also generalized when the end points do not correspond to equilibrium but to stationary states [29]. In the work done by Jayannavar and Sahoo [30], the JE has been verified through the analytical calculation of the work distributions for a charged Brownian particle in a two-dimensional harmonic trap under the action of a uniform magnetic field. Besides, the center of the trap is dragged externally in a known way. This study has been extended to the numerical calculation of the work distributions for a time-dependent magnetic field [31]. The same problem studied in Ref. [30] was considered and generalized in Ref. [32] for an arbitrary time-dependent motion of the minimum in the harmonic trap. It must be noticed that the studies done in works [30–32] are basically related with the explicit solution of the associated overdamped Langevin equation with the electrically charged Brownian particle, though the explicit solution of the complete Langevin equation, including the inertia term, for the classical harmonic oscillator in a constant magnetic field has

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been recently reported by us in Ref. [33]. As far as we know, the explicit solution of the Fokker-Planck (FP) equation associated with this complete Langevin equation has not been reported yet in the literature. However, in the overdamped approximation, the associated FP equation named the Smoluchowski equation admits an explicit solution and this will be part of our task in this work.

Lastly in this work, we also proof the transient work-fluctuation theorem and verify the JE when the arbitrary time-dependent electric field is the responsible for the dragging of the center of the harmonic trap potential, following the proposal in Ref. [32]. This work is then structured as follows: in Sec. II, we give the explicit solution of the Smoluchowski equation by means of an appropriate mathematical transformation allowing this equation be easily solvable. In Sec. III, we obtain the main and marginal fluctuation relations and establish the conditions under which the marginal fluctuation relations reduce to those calculated by Roy and Kumar [17]. In Sec. IV, the transient work-fluctuation theorem and the JE equality are verified. Our conclusions are finally given in Sec. V.

II. LANGEVIN AND SMOLUCHOWSKI EQUATIONS

Let us consider a Brownian particle in a harmonic potential given by $U(\mathbf{r})=\frac{k}{2}|\mathbf{r}|^2$, with k as a constant. The Langevin equation in terms of the velocity $\mathbf{v}$ of a charged particle embedded in a fluid in the presence of an external electromagnetic field (via the Lorentz force) and in the harmonic potential defined above can be written as

$$m\frac{d\mathbf{v}}{dt} = -\gamma\mathbf{v} + \frac{q}{c}\mathbf{v}\times\mathbf{B} + q\mathbf{E}(t) - k\mathbf{r} + \mathbf{A}(t), \tag{1}$$

where $\gamma > 0$ is the friction coefficient, q is the charge of the particle, m is its mass, $\mathbf{B}$ is the uniform magnetic field, $\mathbf{E}(t)$ is a general time-varying electric field, and $\mathbf{A}(t)$ is the fluctuating force which satisfies the properties of Gaussian white noise with zero mean value $\langle A_i(t)\rangle=0$ and a correlation function given by

$$\langle A_i(t)A_j(t')\rangle = 2\lambda\delta_{ij}\delta(t-t'), \tag{2}$$

with $i,j=x,y,z$. λ is a constant which measures the noise intensity and, according to the fluctuation-dissipation theorem, is related with the friction constant by $\lambda=\gamma k_B T$ with k_B as the Boltzmann constant and T as the temperature of the surrounding medium. If the constant magnetic field is assumed to point along the z axis of the Cartesian reference frame, that is, $\mathbf{B}=(0,0,B)$ with B as the modulus of the vector $\mathbf{B}$, then Langevin Eq. (1) can be separated into two independent processes: one takes place along the magnetic field (the z axis) and the other one takes place on the x-y plane, perpendicular to this field. In this work we will focus only on the planar overdamped Langevin equation for which we first define the following vectors on the x-y plane perpendicular to the magnetic field: $\mathbf{x}$ as the position vector, $\mathbf{u}$ as the velocity vector, $\mathbb{A}(t)$ as the fluctuating force vector, such that $\mathbb{A}(t)=(A_x,A_y)$, and $\mathbb{E}(t)$ as the electric field vector. In the overdamped approximation, the required condition to be satisfied by the parameters of Eq. (1) is $\omega^2 \ll \rho^2[1+(\Omega/\rho)^2]^2$, where $\omega^2=k/m$ is the characteristic oscillator frequency, $\rho=\gamma/m$, and $\Omega=qB/mc$ is Larmor's frequency. This condition is equivalent to $km \ll \gamma_e^2$, where $\gamma_e=\gamma(1+C^2)$, $C=qB/c\gamma$ is a dimensionless parameter. In this case γ_e accounts for a redefinition of the friction coefficient γ, which is clearly magnetic field dependent. The planar overdamped Langevin equation reads

$$\frac{d\mathbf{x}}{dt} = -\Lambda\mathbf{x} + \mathbf{b}(t) + \mathbf{G}(t), \tag{3}$$

where

$$\mathbf{b}(t) = qk^{-1}\Lambda\mathbb{E}(t), \quad \mathbf{G}(t) = k^{-1}\Lambda\mathbb{A}(t), \tag{4}$$

and Λ is a 2×2 matrix given by

$$\Lambda = \begin{pmatrix} \tilde{\gamma} & \tilde{\Omega} \\ -\tilde{\Omega} & \tilde{\gamma} \end{pmatrix}, \tag{5}$$

with $\tilde{\gamma}=k/\gamma(1+C^2)=k/\gamma_e$ and $\tilde{\Omega}=kC/\gamma(1+C^2)$. Here the relaxation time is $\tilde{\tau}_r=\tilde{\gamma}^{-1}=\gamma_e/k$, which contains the influence of the magnetic field. Equation (3) is a coupled system of two equations, whose solution in the absence of the electric field has been given by Jayannavar and Sahoo [30] by means of complex numbers, and used to verify the JE for two different dragging protocols. The Smoluchowski equation associated with Eq. (3) requires both the drift D_i and diffusion D_{ij} coefficients, which are easily calculated yielding to [34–36]

$$D_i = -\Lambda_{ij}x_j + b_i, \tag{6}$$

$$D_{ij} = k^{-2}\lambda\Lambda_{ik}\Lambda_{jk}, \tag{7}$$

where Λ_{ij} are the matrix elements of Λ and b_i are the components of vector $\mathbf{b}(t)$. Therefore, the Smoluchowski equation associated with Eq. (3) reads as

$$\frac{\partial P}{\partial t} + \mathbf{b}\cdot\mathbf{grad}_{\mathbf{x}}\, P = \mathbf{div}_{\mathbf{x}}(\Lambda\mathbf{x}P) + \tilde{\lambda}\nabla_{\mathbf{x}}^2 P, \tag{8}$$

subject to the initial condition $P(\mathbf{x},0|\mathbf{x}_0)=\delta(\mathbf{x}-\mathbf{x}_0)$, with $\tilde{\lambda}=\lambda/\gamma^2(1+C^2)$. The explicit solution of Eq. (8) is not easy to calculate due to the coupling term appearing in the first term of the right-hand side of this equation . However, it can be solved explicitly by means of a transformation defined by $\mathbf{X}'=\mathcal{R}(t)(\mathbf{x}-\langle\mathbf{x}\rangle)$, where $\mathcal{R}(t)=e^{\tilde{W}t}$ represents an orthogonal rotation matrix, with

$$\tilde{W} = \begin{pmatrix} 0 & \tilde{\Omega} \\ -\tilde{\Omega} & 0 \end{pmatrix}, \quad \mathcal{R}(t) = \begin{pmatrix} \cos\tilde{\Omega}t & \sin\tilde{\Omega}t \\ -\sin\tilde{\Omega}t & \cos\tilde{\Omega}t \end{pmatrix}, \tag{9}$$

and $\langle\mathbf{x}\rangle$ is the deterministic solution of Eq. (3). Both $\langle\mathbf{x}\rangle$ and the new coordinate $\mathbf{X}'$ satisfy the following differential equations:

$$\frac{d\langle\mathbf{x}\rangle}{dt} = -\Lambda\langle\mathbf{x}\rangle + \mathbf{b}(t), \tag{10}$$

$$\frac{d\mathbf{X}'}{dt} = -\tilde{\gamma}\mathbf{X}' + \mathbf{G}'(t), \tag{11}$$

with $\mathbf{G}'(t)=\mathcal{R}(t)\mathbf{G}(t)$. Equation (11) is easily solved because its mathematical structure is the same as in the ordinary Brownian motion. It is not difficult to show that the drift and diffusion coefficients, in this new representation, are given by

$$D_i' = -\tilde{\gamma}X_i', \tag{12}$$

$$D_{ij}' = k^{-2}\lambda\Lambda_{ik}\Lambda_{jk}. \tag{13}$$

Diffusion coefficient (13) is exactly the same as that given by Eq. (7) because the original noise $\mathbb{A}(t)$ has the same statistical properties of Gaussian white noise than the rotated noise $\mathbf{G}'(t)=\mathcal{R}(t)\mathbf{G}(t)$. Hence, the Smoluchowski equation associated with Langevin Eq. (11), for the transition probability density $P'(\mathbf{X}',t|\mathbf{X}_0')$ of position $\mathbf{X}'$ at time t, conditioned by the initial data $\mathbf{X}'(0)\equiv\mathbf{X}_0'$ at time $t=0$ will be

$$\frac{\partial P'}{\partial t} = \tilde{\gamma}\,\mathbf{div}_{\mathbf{X}'}(\mathbf{X}'P') + \tilde{\lambda}\nabla^2_{\mathbf{X}'}P', \tag{14}$$

subject to the initial condition $P'(\mathbf{X}',0|\mathbf{X}_0')=\delta(\mathbf{X}'-\mathbf{X}_0')$. The solution of Eq. (14) is well known [35–37] and reads

$$P'(\mathbf{X}',t|\mathbf{X}_0') = \frac{k}{2\pi k_BT(1-e^{-2\tilde{\gamma}t})} \times \exp\left\{-\frac{k|\mathbf{X}'-e^{-\tilde{\gamma}t}\mathbf{X}_0'|^2}{2k_BT(1-e^{-2\tilde{\gamma}t})}\right\}. \tag{15}$$

To return to the original variable $\mathbf{x}$, we use the strategy in Ref. [34] to show that the transformation between P' and $P(\mathbf{x},t|\mathbf{x}_0)$ satisfies $P'=P$, and therefore the solution of Smoluchowski Eq. (8) will be given by

$$P(\mathbf{x},t|\mathbf{x}_0) = \frac{k}{2\pi k_BT(1-e^{-2\tilde{\gamma}t})} \times \exp\left(\frac{-k|\mathbf{x}-e^{-\Lambda t}[\overline{\mathbf{b}}(t)+\mathbf{x}_0]|^2}{2k_BT(1-e^{-2\tilde{\gamma}t})}\right), \tag{16}$$

where we have written the deterministic solution as $\langle\mathbf{x}(t)\rangle = e^{-\Lambda t}[\mathbf{x}_0+\overline{\mathbf{b}}(t)]$, with $\overline{\mathbf{b}}(t)=\int_0^t e^{\Lambda s}\mathbf{b}(s)ds$. The joint probability density $f(\mathbf{x},t)$ can be easily calculated from the integral

$$f(\mathbf{x},t) = \int_{\mathbf{x}_0}\delta(\mathbf{x}_0)P(\mathbf{x},t|\mathbf{x}_0)d\mathbf{x}_0. \tag{17}$$

Hence

$$f(\mathbf{x},t) = \frac{k}{2\pi k_BT(1-e^{-2\tilde{\gamma}t})} \times \exp\left(-\frac{k|\mathbf{x}-e^{-\Lambda t}\overline{\mathbf{b}}(t)|^2}{2k_BT(1-e^{-2\tilde{\gamma}t})}\right), \tag{18}$$

where the deterministic solution is $\langle\mathbf{x}\rangle=e^{-\Lambda t}\overline{\mathbf{b}}(t)$.

III. WORK-FLUCTUATION RELATIONS

Inspired in the study done very recently by Roy and Kumar [17], we can also establish three fluctuation relations for a Brownian charged harmonic oscillator in the presence of an electromagnetic field under the conditions established in the previous sections for these fields. The main relation will be given in terms of the joint probability density $f(\mathbf{x},t) = f(x,y,t)$ and the other two are given in terms of the marginal probabilities $f(x,t)$ and $f(y,t)$. Thus from Eq. (18), we obtain a main fluctuation relation given by

$$\frac{f(\mathbf{x},t)}{f(-\mathbf{x},t)} = \exp\left(\frac{2k\mathbf{x}\cdot\langle\mathbf{x}\rangle}{k_BT(1-e^{-2\tilde{\gamma}t})}\right), \tag{19}$$

from which we also can see that

$$\left\langle\exp\left(\frac{-2k\mathbf{x}\cdot\langle\mathbf{x}\rangle}{k_BT(1-e^{-2\tilde{\gamma}t})}\right)\right\rangle = \int_{-\infty}^{+\infty}\exp\left(\frac{-2k\mathbf{x}\cdot\langle\mathbf{x}\rangle}{k_BT(1-e^{-2\tilde{\gamma}t})}\right)f(\mathbf{x},t)d\mathbf{x} = \int_{-\infty}^{+\infty}f(-\mathbf{x},t)d\mathbf{x} = 1. \tag{20}$$

To understand the meaning of Eq. (19), let us analyze the exponent of this equation. To achieve this goal we notice that the time-dependent electric field in Eq. (3) can be associated to the harmonic force $\mathbb{F}=-k(\mathbf{x}-\mathbf{x}^*)$, where the position of the minimum in the trap $\mathbf{x}^*$ can be written as $\mathbf{x}^*(t)=(q/k)\mathbb{E}(t)$. Also this force is derived from the potential $U_C=(k/2)|\mathbf{x}-\mathbf{x}^*|^2$. Now, the minimum in the harmonic trap is dragged by the time-dependent electric field. The mean value $\langle\mathbf{x}\rangle$ in Eq. (19) can be written as

$$\langle\mathbf{x}\rangle = \mathbf{x}^* - e^{-\Lambda t}\mathbf{x}_0^* - e^{-\Lambda t}\int_0^t e^{\Lambda t'}\mathbf{u}^*(t')dt', \tag{21}$$

where $\mathbf{x}_0^*=(q/k)\mathbb{E}(0)$, $\mathbf{u}^*(t)=\dot{\mathbf{x}}^*=(q/k)\dot{\mathbb{E}}(t)$, and Λ is given by Eq. (5).

As a particular case, let us suppose that the electric field is constant and pointing along the x axis, i.e., $\mathbb{E}=(E_x,0)$, with E_x as a constant. In this case the potential's minimum is fixed in the position $\mathbf{x}^*\equiv(x^*,y^*)=(q/k)(E_x,0)$ and therefore

$$\langle x\rangle = x^*(1-e^{-\tilde{\gamma}t}\cos\tilde{\Omega}t), \quad \langle y\rangle = -x^*e^{-\tilde{\gamma}t}\sin\tilde{\Omega}t. \tag{22}$$

It is worth noticing that the deterministic solution [Eq. (22)] corresponds to a spiral in the x-y plane, as it should be, and the trajectory characteristics depends on both the electric and magnetic fields. Consequently, the term $k\mathbf{x}\cdot\langle\mathbf{x}\rangle$, which represents the projection of the harmonic force $k\mathbf{x}$ along the deterministic trajectory $\langle\mathbf{x}\rangle$, represents the work done by this force along that trajectory. However, this harmonic force is coupled to the electric field and therefore this work can be written as

$$W(t) = \frac{2qE_x}{1-e^{-2\tilde{\gamma}t}}\{x(1-e^{-\tilde{\gamma}t}\cos\tilde{\Omega}t) - ye^{-\tilde{\gamma}t}\sin\tilde{\Omega}t\}. \tag{23}$$

On the other hand, the second term of Eq. (21) is $\mathcal{R}^{-1}(t)\mathbf{x}^* = (q/k)\mathcal{R}^{-1}(t)(E_x,0)$ which is a time-dependent rotation of the electric field which we define as $\mathbf{E}'=\mathcal{R}^{-1}(t)(E_x,0)$. In a rotating reference frame, the electric field has two components, namely, $E_x'(t)=E_x\cos\tilde{\Omega}t$ and $E_y'(t)=E_x\sin\tilde{\Omega}t$, where $\tilde{\Omega}$ measures the rotation frequency, and thus $qxE_x\cos\tilde{\Omega}t$

$=qE_x'x$ and $qyE_x \sin \tilde{\Omega}t = qE_y'y$ are the works done by these components along the x and y axes, respectively. These quantities multiplied by $e^{-\tilde{\gamma}t}$ are then the dissipation of these works due to the friction coefficient $\tilde{\gamma}$. Therefore Eq. (23) can be interpreted as the work done by the external electric field and it takes into account that the displacement of the Brownian particle occurs in the x and y directions, as seen from a fixed reference frame.

When the electric field is a general time-dependent vector, it can be considered as the responsible for the dragging of the potential's minimum in an arbitrary way. In this case we assume that, at the beginning of the movement at $t=0$, the potential's minimum is located at the origin of coordinates such that $\mathbf{x}_0^*=0$. For $t>0$ we take $\mathbb{E}(t)=(\mathcal{E},\mathcal{E})\phi(t)$, where $\mathcal{E}$ is a constant electric field and $\phi(t)$ is an arbitrary scalar dimensionless function of time. In this case, the dragging velocity reads as $\dot{\mathbf{x}}^*=\mathbf{u}^*=(q/k)\mathcal{E}\dot{\phi}(t)\mathbb{I}$ and $\mathbb{I}=(1,1)$. Under these conditions it is easy to show that

$$W(t) = \frac{2q\mathcal{E}}{1-e^{-2\tilde{\gamma}t}}\{x[\phi(t)-I_x(t)] + y[\phi(t)-I_y(t)]\}, \quad (24)$$

where I_x and I_y are the components of the vectorial function,

$$\mathbf{I}(t) = \int_0^t e^{\tilde{\gamma}(t-t')}\mathcal{R}^{-1}(t)\mathcal{R}(t')\dot{\phi}(t')\mathbb{I}dt'. \quad (25)$$

Similarly the terms $q\mathcal{E}x$ and $q\mathcal{E}y$ correspond to the works done by the constant electric field $\mathcal{E}$ along the x and y axes, respectively, and $q\mathcal{E}xI_x(t)$, $q\mathcal{E}yI_y(t)$ contain the dissipation contribution of those terms. In this general case the fluctuation relation given by (19) can be written in terms of the total dissipative work done by the external electric field as

$$\frac{f(\mathbf{x},t)}{f(-\mathbf{x},t)} = e^{W(t)/k_BT}. \quad (26)$$

This relation is consistent with the fluctuation theorems [18,19], which state that the probability to observe a given trajectory at $(\mathbf{x},t)$ during the forward process, and the probability to observe its backward counterpart at $(-\mathbf{x},t)$, is essentially related with the work done by the externally applied time-dependent electric field.

The fluctuation relations for the marginal probability densities $f(x,t)$ and $f(y,t)$ can be calculated easily from Eq. (18), through the integrals $f(x,t)=\int f(x,y,t)dy$ and $f(y,t)=\int f(x,y,t)dx$, which after integration we obtain

$$f(x,t) = \sqrt{\frac{k}{2\pi k_BT(1-e^{-2\tilde{\gamma}t})}} \times \exp\left(-\frac{k(x-\langle x\rangle)^2}{2k_BT(1-e^{-2\tilde{\gamma}t})}\right), \quad (27)$$

$$f(y,t) = \sqrt{\frac{k}{2\pi k_BT(1-e^{-2\tilde{\gamma}t})}} \times \exp\left(-\frac{k(y-\langle y\rangle)^2}{2k_BT(1-e^{-2\tilde{\gamma}t})}\right). \quad (28)$$

The fluctuation relation for both probabilities are established as

$$\frac{f(x,t)}{f(-x,t)} = \exp\left(\frac{2kx\langle x\rangle}{k_BT(1-e^{-2\tilde{\gamma}t})}\right), \quad (29)$$

$$\frac{f(y,t)}{f(-y,t)} = \exp\left(\frac{2ky\langle y\rangle}{k_BT(1-e^{-2\tilde{\gamma}t})}\right), \quad (30)$$

where $\langle x\rangle$ and $\langle y\rangle$ are the components of the vector $\langle \mathbf{x}\rangle$. Using the normalization condition of the probability densities, we also obtain other useful relations, namely,

$$\left\langle \exp\left(\frac{-2kx\langle x\rangle}{k_BT(1-e^{-2\tilde{\gamma}t})}\right)\right\rangle = \int_{-\infty}^{+\infty} \exp\left(\frac{-2kx\langle x\rangle}{k_BT(1-e^{-2\tilde{\gamma}t})}\right)f(x,t)dx = \int_{-\infty}^{+\infty} f(-x,t)dx = 1, \quad (31)$$

$$\left\langle \exp\left(\frac{-2ky\langle y\rangle}{k_BT(1-e^{-2\tilde{\gamma}t})}\right)\right\rangle = \int_{-\infty}^{+\infty} \exp\left(\frac{-2ky\langle y\rangle}{k_BT(1-e^{-2\tilde{\gamma}t})}\right)f(y,t)dy = \int_{-\infty}^{+\infty} f(-y,t)dy = 1. \quad (32)$$

The physical meaning of Eqs. (29) and (30) is similar to that displayed in relation to Eq. (19). In particular, in the absence of the harmonic force and also assuming a constant electric field pointing along the x axis, our theoretical results reduce to the fluctuation relations recently established by Roy and Kumar [17]. For $k=0$, we show in Appendix that marginal fluctuation relation (29) reduces to

$$\frac{f(x,t)}{f(-x,t)} = \exp\left(\frac{-eEx}{k_BT}\right), \quad (33)$$

which is the same barotropic-type fluctuation relation established by Roy and Kumar [17]. Also Eq. (30) reduces to

$$\frac{f(y,t)}{f(-y,t)} = \exp\left(\frac{-e^2EBy}{k_BTc\gamma}\right), \quad (34)$$

which was named as the transverse fluctuation relation or Hall-fluctuation theorem. As discussed above, the argument of the exponential in Eq. (30) is essentially related with the work done, along the y axis, by the component E_y' of the rotated electric field, which reduces to the expression given by Eq. (34) when $k=0$ [see also Eq. (A9) of Appendix]. The marginal probability densities $f(x,t)$ and $f(y,t)$, in this limiting case, are explicitly given in Appendix. As mentioned by Roy and Kumar [17], the interpretation of those relations can be expressed as follows: though the system is always far from thermodynamic equilibrium (as the electron continuously dissipates energy in the environment), it reaches a mechanical equilibrium asymptotically under the combined effect (forcing) of the electromagnetic fields and the viscous drag acting in opposition.

IV. WORK-FLUCTUATION THEOREM AND JARZYNSKI EQUALITY

In a similar way as studied in Refs. [30,33] to verify the transient work-fluctuation theorem and Jarzynski equality

[11], here we also consider that the dragging of the center of the harmonic trap potential, $U_C=(k/2)|\mathbf{x}-\mathbf{x}^*|^2$, is driven by the presence of an arbitrary time-dependent electric field. Thus, as considered in Sec. III $\mathbf{x}^*(t)=(q/k)\mathbb{E}(t)$, again with the initial condition $\mathbf{x}_0^*=(q/k)\mathbb{E}(0)=0$. We first calculate the total work done on the system due to the harmonic force $F=-k(\mathbf{x}-\mathbf{x}^*)$, which is given by [8,30,33]

$$W_{tot}=-k\int_0^\tau(\mathbf{x}-\mathbf{x}^*)\cdot\mathbf{u}^*dt. \tag{35}$$

Following exactly the same algebraic steps in Ref. [32], we immediately conclude that the total work mean value reads as

$$\langle W_{tot}\rangle=k\int_0^\tau dt\int_0^t e^{-\tilde{\gamma}(t-t')}\mathbf{U}^*(t)\cdot\mathbf{U}^*(t')dt', \tag{36}$$

where $\mathbf{U}^*(t)=\mathcal{R}(t)\mathbf{u}^*(t)$. The variance $V=\langle W_{tot}^2\rangle-\langle W_{tot}\rangle^2$ is also

$$V=2k_BTk\int_0^\tau dt\int_0^t e^{-\tilde{\gamma}(t-t')}\mathbf{U}^*(t)\cdot\mathbf{U}^*(t')dt'. \tag{37}$$

Therefore $V=2k_BT\langle W_{tot}\rangle$ and the transient work-fluctuation theorem for this total work is coming up, that is $P(-W_{tot})=e^{-W_{tot}/k_BT}P(W_{tot})$. From the Jarzynski equality we have $\langle e^{-W_{tot}/k_BT}\rangle=e^{-\Delta F/k_BT}=1$, and therefore $\Delta F=0$, consistent with the Bohr–van Leeuwen theorem on the absence of orbital diamagnetism in a classical system of charged particles in thermodynamic equilibrium [38].

V. CONCLUDING REMARKS

We have established the fluctuation relation given by Eq. (19) for a system of two-dimensional noninteracting charged harmonic oscillators under the action of an electromagnetic field in the presence of a dissipative environment at temperature T in the high friction limit. This relation has been achieved through the explicit solution of Smoluchowski Eq. (8) for the joint probability density given by Eq. (18). As we have shown, this fluctuation relation is related with the work done by the externally time-dependent electric field responsible for the dragging of the potential's minimum from $t=0$ to time t [see Eq. (26)]. In this sense fluctuation relation (19) is consistent with the fluctuation theorems established by Bochkov and Kuzovlev [18] and Horowitz and Jarzynski [19]. The two marginal fluctuation relations given by Eqs. (29) and (30) have a similar physical interpretation than main fluctuation relation (19). In the absence of the harmonic force and assuming that the electric field is considered as a constant and perpendicular to the magnetic field, marginal fluctuation relations (29) and (30) reduce exactly to those given by Eqs. (33) and (34), respectively, which were recently calculated by Roy and Kumar [17] as expected.

On the other hand, in Refs. [30,32] it has been shown that, for the same system of noninteracting harmonic oscillators in the harmonic potential U_C and in the presence of the magnetic field only, the transient work-fluctuation theorem holds, and also $\Delta F=0$ as a consequence of the Jarzynski equality which is consistent with the Bohr–van Leeuwen theorem [38]. For the same potential U_C and in the presence of an electromagnetic field, we have shown that, for an arbitrary time-dependent electric field responsible for the dragging of the center of the harmonic trap potential, the work-fluctuation theorem and $\Delta F=0$ are also satisfied.

Finally, we would like to comment here that the mathematical treatment done in this work illustrates how the rotation matrices involved in the procedure simplify in an elegant way the algebraic steps when there is a magnetic field involved in the problem. This procedure allows us to calculate explicitly the joint probability density $f(x,y,t)$ for the Smoluchowski equation which may be difficult to obtain by other methods, and it generalizes the treatment performed by Roy and Kumar [17] for the marginal probabilities.

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APPENDIX: FLUCTUATION RELATIONS OF ROY AND KUMAR

To obtain the fluctuation relations of Roy and Kumar, we first write the marginal fluctuation relations as required by Eq. (23). In this case, Eqs. (29) and (30) will be given, respectively, by

$$\frac{f(x,t)}{f(-x,t)}=\exp\left(\frac{2qEx(1-e^{-\tilde{\gamma}t}\cos\tilde{\Omega}t)}{k_BT(1-e^{-2\tilde{\gamma}t})}\right), \tag{A1}$$

$$\frac{f(y,t)}{f(-y,t)}=\exp\left(\frac{2qEye^{-\tilde{\gamma}t}\sin\tilde{\Omega}t}{k_BT(1-e^{-2\tilde{\gamma}t})}\right), \tag{A2}$$

where $E_x=E$. To evaluate these equations for $k=0$, we first multiply and divide by the factor k the arguments of the exponential. By taking into account that $1-e^{-x}=-\sum_{n=1}^\infty(-1)^n x^n/n!$, we thus have

$$\begin{aligned}\frac{k}{(1-e^{-2\tilde{\gamma}t})}&=\left[-\sum_{n=1}^\infty(-1)^n\left(\frac{2t}{\gamma(1+C^2)}\right)^n\frac{k^{(n-1)}}{n!}\right]^{-1}\\&=\left[\frac{2t}{\gamma(1+C^2)}-\sum_{n=2}^\infty(-1)^n\left(\frac{2t}{\gamma(1+C^2)}\right)^n\frac{k^{(n-1)}}{n!}\right]^{-1}\\&=\left(\frac{2t}{\gamma(1+C^2)}\right)^{-1}\end{aligned} \tag{A3}$$

for $k=0$. Also

$$\cos(\tilde{\Omega}t)=1+\sum_{n=1}^\infty\left(\frac{Ct}{\gamma(1+C^2)}\right)^{2n}\frac{k^{2n}}{(2n)!}, \tag{A4}$$

$$\sin(\tilde{\Omega}t) = \frac{Ckt}{\gamma(1+C^2)} + \sum_{n=1}^{\infty}(-1)^{2n+1}\left(\frac{Ct}{\gamma(1+C^2)}\right)^{2n+1}\frac{k^{2n+1}}{(2n+1)!}. \tag{A5}$$

After some algebra we have

$$\frac{1}{k}(1-e^{-\tilde{\gamma}t}\cos\tilde{\Omega}t) = \left(\frac{t}{\gamma(1+C^2)}\right), \tag{A6}$$

$$\frac{1}{k}e^{-\tilde{\gamma}t}\cos\tilde{\Omega}t = \left(\frac{Ct}{\gamma(1+C^2)}\right) \tag{A7}$$

if $k=0$. If we now make $q=-e$, thus $C=qB/c\gamma=-eB/c\gamma$, and if we define $\bar{C}=eB/c\gamma$, then we can conclude exactly that when $k=0$

$$\frac{2qEx(1-e^{-\tilde{\gamma}t}\cos\tilde{\Omega}t)}{k_BT(1-e^{-2\tilde{\gamma}t})} = -\frac{eEx}{k_BT}, \tag{A8}$$

$$\frac{2qEye^{-\tilde{\gamma}t}\sin\tilde{\Omega}t}{k_BT(1-e^{-2\tilde{\gamma}t})} = -\frac{eEy\bar{C}}{k_BT}, \tag{A9}$$

and therefore

$$\frac{f(x,t)}{f(-x,t)} = \exp\left(\frac{-eEx}{k_BT}\right), \tag{A10}$$

$$\frac{f(y,t)}{f(-y,t)} = \exp\left(\frac{-e^2EBy}{k_BTc\gamma}\right), \tag{A11}$$

where the marginal probability densities $f(x,t)$ and $f(y,t)$ are given by

$$f(x,t) = \sqrt{\frac{1+\bar{C}^2}{4\pi Dt}}\exp\left\{-\frac{1+\bar{C}^2}{4Dt}(x+\langle x\rangle)^2\right\}, \tag{A12}$$

$$f(y,t) = \sqrt{\frac{1+\bar{C}^2}{4\pi Dt}}\exp\left\{-\frac{1+\bar{C}^2}{4Dt}(y+\langle y\rangle)^2\right\}, \tag{A13}$$

with $\langle x\rangle=-\alpha t/(1+\bar{C}^2)$ and $\langle y\rangle=-\alpha\bar{C}t/(1+\bar{C}^2)$ with $\alpha=eE/\gamma$. Equations (A10)–(A13) are exactly the same expressions calculated by Roy and Kumar [17].

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